Basic trigonometric identities

Remark: You should memorize all numerated formulas.

№1. Prove the so called *Pythagorean trigonometric identity*:

$$\sin^2\varphi + \cos^2\varphi = 1. \tag{1}$$

№2. Also prove this handy identity[1]:

$$\tan\varphi \cdot \cot\varphi = 1. \tag{2}$$

№3. Can these equalities hold simultaneously?

1. $\sin\alpha = \dfrac{1}{3}, \quad \cos\alpha = \dfrac{2\sqrt{2}}{3};$
2. $\tan\alpha = 5, \quad \cot\alpha = 0.2;$
3. $\sin\alpha = \dfrac{a}{\sqrt{a^2+b^2}}, \quad \cos\varphi = \dfrac{b}{\sqrt{a^2+b^2}};$
4. $\sin\alpha = \dfrac{2\sqrt{4+2a}}{a+4}, \quad \cos\varphi = -\dfrac{a}{a+4}.$

№4. Now prove these two identities:

$$1 + \tan^2\varphi = \frac{1}{\cos^2\varphi}, \tag{3}$$

$$1 + \cot^2\varphi = \frac{1}{\sin^2\varphi}. \tag{4}$$

№5. Can these equalities hold simultaneously?

1. $\cos\alpha = \dfrac{2}{3}, \quad \tan\alpha = -\dfrac{1}{3};$
2. $\sin\alpha = -\dfrac{2}{7}, \quad \cot\alpha = \dfrac{3\sqrt{5}}{2}.$

№6. Compute the value of trigonometric functions of angle β:

1. $\sin\beta = -\dfrac{1}{4};$
2. $\tan\alpha = -3, \quad \dfrac{\pi}{2} < \beta < \pi.$

№7. Prove the following identities:

1. $\sin^4\alpha\cos^2\alpha + \sin^2\alpha\cos^4\alpha = \sin^2\alpha\cos^2\alpha;$
2. $\cot^2\alpha - \cos^2\alpha = \cot^2\alpha\cos^2\alpha;$
3. $\left(\tan^2\alpha - \sin^2\alpha\right)\cdot\dfrac{\cot^2\alpha}{\sin^2\alpha} = 1;$
4. $2\left(\sin^6\alpha + \cos^6\alpha\right) - 3\left(\sin^4\alpha + \cos^4\alpha\right) = -1.$

[1] In post-Soviet counries $\tan x$ is usually denoted as $\operatorname{tg} x$ and $\operatorname{ctg} x$ stands for $\cot x$.